Appendix A

Gaussian Integrals

While the Gaussian integral is discussed in a number of mathematical textbooks, we briefly summarize some key features here for readers who have not yet encountered it. To evaluate the integral

$$I = \int_{-\infty}^{\infty} e^{-ax^2} dx$$

it is useful to multiply the integral by itself, taking care to introduce a second dummy variable of integration:

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-ax^2} dx \int_{-\infty}^{\infty} e^{-ay^2} dy \\ &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; e^{-a\left(x^2+y^2\right)} \end{aligned}$$

The form of this integral suggests a transformation to polar coordinates since $r^2=x^2+y^2$. Replacing dxdy with rdrdθ we get

$$I^2 = \int_0^{2\pi} d\theta \int_0^{\infty} rdr \; e^{-ar^2}$$

The lack of θ dependence makes the first integral trivial (2π). The second integral is not much harder:

$$I^2 = \left(2\pi\right)\left[\frac{-1}{2a} e^{-ar^2}\right]_0^{\infty} = \frac{\pi}{a}$$

Thus we arrive at the key result,

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

From this result, a number of others follow. For instance, since the integrand is symmetric

$$\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

If we return to the previous equation and differentiate both sides with respect to the variable a (and multiply by -1)

$$\int_{-\infty}^\infty x^2 e^{-ax^2}dx = \frac{1}{2}\pi^{1/2}a^{-3/2}$$

Repeated differentiation gives the results for integrands involving all of the even powers of x.

We can also invoke the symmetry of the Gaussian function to conclude that

$$\int_{-\infty}^\infty x^n e^{-ax^2}dx = 0 \qquad \text{(n odd)}$$

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