Module M1

Mechanics of Planar and Three Dimensional Rotations of Rigid Bodies

Prerequisite: Module C1

This module requires you to read a textbook such as Fowles and Cassiday on material relevant to the following topics.

Topics:

Center of mass
Moment of inertia about a fixed axis
Perpendicular and parallel axis theorems
Radius of gyration
Physical pendulum
Laminar motion (angular momentum, energy considerations, rolling without slipping)
Collisions and the baseball bat theorem

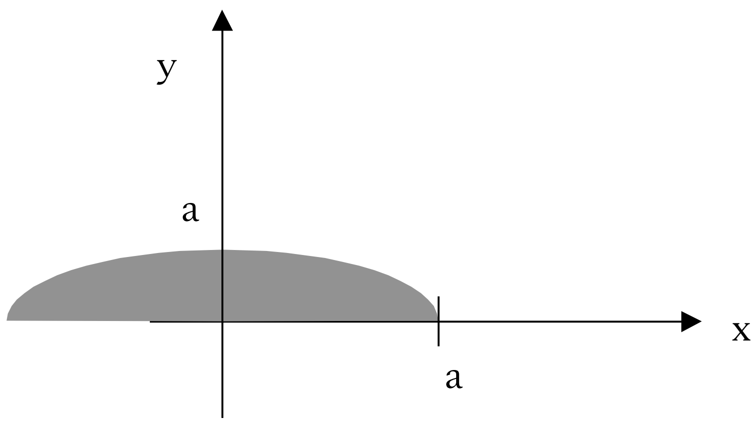
Rotations about an arbitrary axis
Moment of inertia tensor & its use in determining angular momentum and kinetic energy
Principal axes
Euler's equations
Free rotation
Eulerian angles
Motion of a top

References:
Fowles and Cassiday Analytical Mechanics 7th ed (Thomson/Brooks/Cole 2005) Chapter 8, Chapter 9 (sections 1-7)

Thornton and Marion Classical Dynamics of Particles and Systems 5th ed (Thomson/Brooks/Cole 2004) Chapter 11

Assignment: Note that many of these problems are greatly influenced by (and in many cases virtually identical to) Fowles and Cassiday

1. Consider a uniform thin, flat object of mass m whose boundaries can be described by the equations $y=a- x^2/a$ and the x axis.



a. Determine the center of mass of this object.
b. Determine its moment of inertia about the y axis.

2. A thin rod of length b has a nonuniform density, expressed as mass per unit length, given by $\lambda = c\sqrt{x}$, where c is a constant and x is the distance from the left end of the rod. Determine the center of mass of this rod and the moment of inertia about its left end. Express your result in terms of b and the total mass, m, of the rod.

3. Pendulum 1 consists of a point mass, M, attached to the end of a massless rod of length "a". Pendulum 2 (a more realistic description of a real pendulum) consists of a solid spherical bob of mass M-m and radius b, attached to a rod of mass m and length a-b. Notice that the distance from the pivot point to the center of the bob is, in both cases, "a", and the total mass is, in both cases, M.
a. Calculate the ratio of the period of Pendulum 2 to Pendulum 1. Express your result in terms of $\alpha=m/M$ and $\beta=b/a$.
b. Determine a numerical value for this ratio when a=1.10m, b=8.0cm, m=50g and M=500g.

4. A thin, uniform disk of radius b pivots about a point on its edge. It can oscillate as a physical pendulum with the rotation axis either perpendicular to the disk and passing through the pivot point or in the plane of the disk, passing through the pivot point and tangent to the disk. Calculate the period of oscillation in each of these two cases.

5. A uniform board of length L and mass M is supported at each end. At t=0, the right end is suddenly released.
a. Right after the release, what is the angular acceleration of the board?
b. Right after the release, what is the linear acceleration of the right end of the board?
c. Right after the release, what is the normal force exerted by the left support on the board?

6. Consider the nonuniform rod described in problem 2. Suppose it is lying on a frictionless surface and the right end (the heavy end) is given a sharp kick (perpendicular to the rod). Find the location of the point about which the plank begins to rotate.

7. As shown in Fowles and Cassiday, the differential equation of motion for a physical pendulum is given by

$$I\ddot{\theta} + mgl\sin\theta = 0$$

An exact expression for the period of motion is given by

$$T = 4\sqrt{\frac{I}{mgl}}K(q)$$

where K(q) is the complete elliptic integral of the first kind (see, for instance, Handbook of Mathematical Functions by Abramowitz and Stegun), $q=\sin^2(\theta_o/2)$ and θ_o is the amplitude of the oscillation. This expression is in contrast to the one valid for small amplitude oscillations,

$$T_o = 2\pi\sqrt{\frac{I}{mgl}}$$

Consider a physical pendulum released from rest at $\theta_o=100^o$.

a. Use a math reference or appropriate software to look up the value of K(q) and from that determine the period of motion. Express your answer as a number times T_o.

b. The complete elliptic integral of the first kind is defined as $K(q) = \int_0^{\pi/2} \left(1 - q\sin^2\theta\right)^{-1/2} d\theta$. Use this expression and a spreadsheet routine to numerically determine the value of K(q) for the given conditions. What do you need to do to get this value to agree with the tabular value to within 1%? For this part, turn in a discussion of your results and a print out of the first page of your spreadsheet.

c. Use the original differential equation of motion, and integrate that numerically using a spreadsheet to determine the period of motion (in terms of T_o). What do you need to do to make this result in agreement with the value in part (a) to within 1%? For this part, turn in work showing how you rewrote the differential equation to make it suitable for spreadsheet integration, a discussion of your results, and the first page of your spreadsheet.

8. A uniform, thin sheet of cardboard, 8.5 inches by 11 inches, has mass M. Taking the cardboard as lying in the positive quadrant of the xy plane with one corner at the origin and the x axis lying along a side of length 8.5 inches,
a. determine the entries in the moment of inertia tensor
b. determine the principle axes for this lamina.
c. determine the moment of inertia about the line passing through the origin and the point x=8.5 inches, y=11 inches, and z=0 (i.e., a diagonal of the cardboard).
d. determine the angular momentum if the plate spins about the axis described in part (c) at a constant rate ω.
e. determine the kinetic energy under the conditions described in part (d).
f. determine if the angular momentum found in (d) is constant in time.

9. Consider a uniform brick of mass 2.0kg and dimensions 6.0cmx10.0cmx20.0cm. Suppose it is spinning about a body diagonal at a constant rate of 1 revolution per second. Taking the origin of your coordinate system to be at the center of the brick,
a. find the kinetic energy of the brick.
b. find the angle between the angular momentum and the angular velocity.
c. find the magnitude of the torque required to maintain this motion.

10. A thin uniform rod of mass m and length L rotates about a fixed axis passing through the center of the rod and making a constant angle α with the rod. Prove the following:
a. The angular momentum about the center of the rod is perpendicular to the rod.
b. The magnitude of the angular momentum about the center of the rod is $(\sin\alpha) mL^2\omega/12$.
c. The angular momentum about the center of the rod is not constant.
d. The torque required to maintain this motion is perpendicular to the rod, to the angular momentum, and to the angular velocity.
e. The magnitude of the torque required to maintain this motion is $\frac{mL^2\omega^2 \sin 2\alpha}{24}$.

11. Consider a rigid body rotating in the absence of any torque. Show that Euler's equations imply that the magnitude of the angular momentum is constant, as is the rotational kinetic energy. Hint: Starting from

$$N_1 = I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2)$$
$$N_2 = I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3)$$
$$N_3 = I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1)$$

multiply the first equation by $I_1\omega_1$, the second by $I_2\omega_2$, and the third by $I_3\omega_3$. Add these equations up and you will be close to showing L^2 is constant. A similar process works for the kinetic energy, but each equation is multiplied by ω instead. In this notation, 1, 2, and 3 refer to the 3 principal axes of the object.

12. Using Euler's equations, show that if an object experiences a torque that is always perpendicular to its angular momentum, then the magnitude of its angular momentum does not change.

13. A uniform block of length b, width b, and thickness b/2 is tossed in the air, with the result being (ignoring air resistance) it rotates with no torque at an angle of 30^o with respect to the symmetry axis of the block. The rotational period is T_o.
a. Determine the period of precession of the axis of rotation about the symmetry axis.
b. Determine the period of wobble of the symmetry axis about the angular momentum vector.

14. Suppose a Frisbee is tossed into the air with a wobble and it experiences a torque due to air resistance of the form $-c\vec{\omega}$. Show that
a. the component of the angular velocity in the direction of the symmetry axis falls off exponentially in time.
b. if the moment of inertia about the symmetry axis is greater than the other two moments of inertia (as is the case for a Frisbee), then the angle between the symmetry axis and $\vec{\omega}$ decreases over time (i.e., the wobble decreases).

15. Consider a block of uniform density and of total mass m. Its dimensions are a x 2a x 4a.
a. The block is to be rotated at constant angular speed ω_o about its body diagonal. Use Euler's equations to determine the torque that must be applied.
b. Suppose the torque is abruptly decreased to zero at time t=0. Use a spreadsheet program or other mathematical software to produce plots of ω_1, ω_2, and ω_3 as a function of time. As a check on your numerical integration, calculate the kinetic energy at each step to ensure it remains constant.

MODULE M1

Name: Term:

Problem	Max	1st	2nd	Final
1	5			
2	5			
3	5			
4	5			
5	5			
6	5			
7	10			
8	10			
9	5			
10	5			
11	5			
12	5			
13	5			
14	5			
15	10			
TOTAL	90			