

Module NED1: Non-Equilibrium Dynamics I

Prerequisites: Modules C1 and C2

A few of the early problems in this unit are among the easiest in this course. They are designed to get you to work through some of the simple algebra on your own.

References:

DeSerio, Robert. "Chaotic pendulum: The complete attractor". American Journal of Physics -- March 2003 -- Volume 71, Issue 3, pp. 250-257.

J. E. Berger and G. Nunes, Jr. "A mechanical Duffing oscillator for the undergraduate laboratory." American Journal of Physics volume 65 Number 9 pages 841-846 (1997).

Addison, Paul S. Fractals and Chaos: An Illustrated Course. Institute of Physics (1997).

The EMU library has numerous other books on fractals and chaos.

Introduction:

A non-linear system can sometimes give rise to seemingly unpredictable results even though the equations governing the evolution of the system are apparently fully deterministic. A system that behaves in this way is described as chaotic. The purpose of this module is to introduce you to a few simple models that are chaotic. It is not intended to be a substitute for a full course on chaos. If you are interested in further studies, browse the shelves of the EMU library—you'll find a wide selection of books there.

A Simple Model:

One of the simplest models exhibiting chaos is actually a model of growth and death in a population of a biological species. Thus we will first digress a bit into the field of population ecology. While I am not aware of physics applications of this model, we will see that it does share features with our next model (which is explicitly physics-based). We begin by defining the carrying capacity, N_c , of a species in a given habitat as being the maximum number of members of the species the habitat can reasonably support, based on resource limitations. That is, the food and shelter available are going to limit the size of a population. If the total number of members of the species is N , then I will define the population fraction, x , as

$$x = \frac{N}{N_c} . \quad (1)$$

We would ordinarily expect x to range between 0 and 1. If we assume the species reproduces at regular intervals (say, for instance, once a year), then we would expect the population in interval $n+1$ to depend on the population in interval n . In particular, the more species members there are (the greater x is) then the more offspring that will be produced, so x_{n+1} should be proportional to x_n . At the same time, the closer the population is to its carrying capacity (the closer x is to 1), then the more species members that will die due to limited resources. Thus we also expect that x_{n+1} should be proportional to $1-x_n$. Putting this together, we have

$$x_{n+1} = kx_n(1 - x_n) . \quad (2)$$

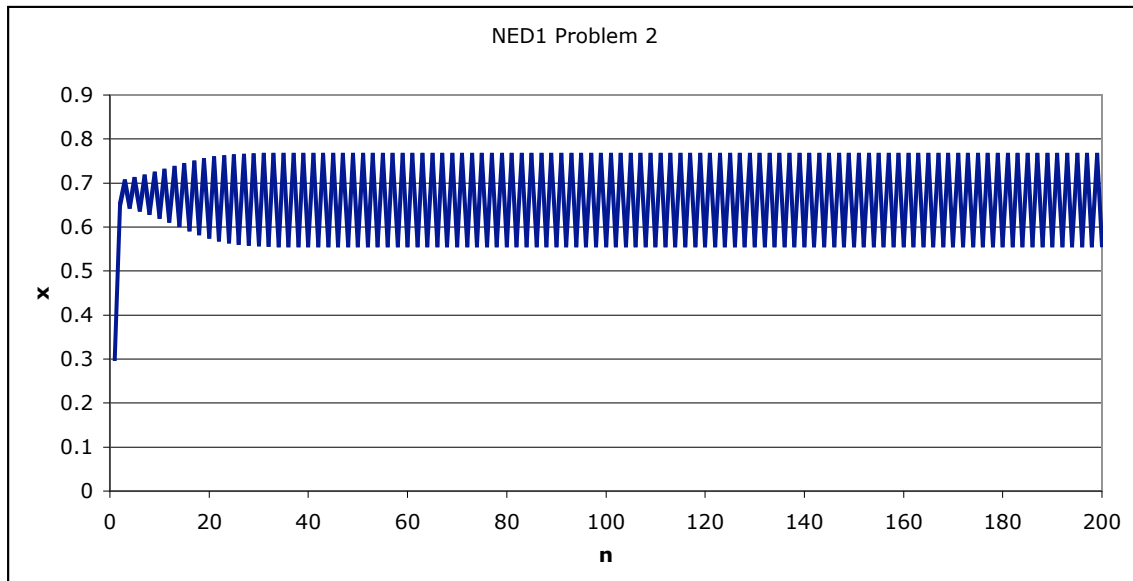
This equation is a finite difference equation, a cousin to the differential equation. Rather than describing the evolution of x in terms of a continuous time variable, t , it describes the evolution of x in terms of discrete time steps, n .

Given our definition of x , the biologically acceptable values range from 0 to 1. You will show below that this means k should be less than or equal to 4.

Problems:

1. Let $f(x)=x(1-x)$. Using calculus, determine the maximum value of $f(x)$. From this result, explain why the value of k in equation (2) should not exceed 4.

2. Set up an Excel spreadsheet to calculate a sequence of $\{x_n\}$ values. Set k as a variable at the top of the spreadsheet since you will be wanting to change it to explore different possibilities. Calculate the first 200 terms in the series and plot x_n vs n . I used the scatter plot format with no data point markers and non-smoothed lines. Take the first value of x to be 0.3. Here is my result for $k=3.1$.



As you format the graph, try to maximize the horizontal spread so that you can see the details clearly.

- Submit plots for $k=2, 2.5, 3.2, 3.5,$ and 3.8 .
- What patterns do you notice for each of these values of k ? You've calculated up to $n=200$. For each of these k values, do you think it is possible to predict the value of x when $n=300$? Make sure to explain your logic (then test it out if you want by extending the calculation to $n=300$).

3. A single fixed point can be defined as a value x_s such that $x_s = kx_s(1 - x_s)$. You should have discovered in the previous problem that when $k=2$, $x_s=0.5$ is a single fixed point.

a. Show that for any value of k , $x=0$ is always a fixed point.

b. Show that $x = 1 - \frac{1}{k}$ is also a fixed point. Given the constraints on x and k discussed above, for what range of values of k is this fixed point relevant?

c. Show that your result in (b) correctly predicts the single fixed points for the $k=2$ and $k=2.5$ cases you looked at in problem 2.

4. When $k=3.2$, you should have noticed the values of x settling down to an oscillation between two fixed points. We'll call those a fixed point pair and label them x_p . Fixed point pairs can be determined by finding a value of x_n that satisfies the following equations:

$$x_{n+1} = kx_n(1 - x_n)$$

$$x_{n+2} = kx_{n+1}(1 - x_{n+1})$$

$$x_n = x_{n+2}$$

When you combine these three equations into a single equation for x_n , you will get a 4th degree polynomial. However, all terms will have at least one factor of x_n , so the polynomial can be reduced to a 3rd degree polynomial. This is another way of saying that $x_n=0$ satisfies all of the constraints above. Likewise, you should be able to convince yourself that $x_n=x_s$ should also be a solution to that system of equations. This allows you to factor the 3rd degree polynomial into the product between a linear polynomial and a quadratic polynomial. The two roots of the quadratic polynomial will give you the two members of the fixed point pair.

a. Find an analytic expression for these roots and compare to your results for the $k=3.2$ case that you studied in problem 2. If you don't have agreement, rework the algebra until you do.

b. What is the minimum value of k allowed in order for your expression for the fixed point pairs to yield two real roots?

c. Verify your conclusion in (b) by choosing a value of k close to but slightly greater than the cutoff in (b). Run the Excel spreadsheet for this value and compare the fixed point pair to the values you calculate using your result from (a).

Comment: In classical mechanics we talk of equilibrium positions, stable equilibrium, and unstable equilibrium. An object is in mechanical equilibrium if the net force on it is zero. This means if we place the object at rest precisely at the equilibrium point, it will not move away from that point. If we place the object close to but not at the equilibrium point, we can determine stability. The object is in stable equilibrium if it has a tendency to move towards the equilibrium point under these circumstances. If it moves away, it is in unstable equilibrium. These same concepts apply to the population model we have been looking at thus far. Single fixed points are equilibrium points. A fixed point pair can be thought of as an equilibrium orbit. It turns out that as k increases from 0, at first there is just one fixed point ($x=0$) and it is stable. When we reach $k=1$, $x=1-1/k$ becomes a viable fixed point and it becomes stable while the $x=0$ point becomes unstable. When we reach $k=3$, the $1-1/k$ fixed point becomes unstable while the fixed point pair becomes a stable orbit.

5. We have looked at single fixed points and orbits containing a pair of fixed points. Vary k to see if you can locate orbits of 3 or more fixed points. Note that if you reflect back on what you have learned so far, you should know on what range of k values to focus your attention. Try to find several other orbit types, if possible.

6. For some values of k , there is no discernable pattern. Under these cases, the system is chaotic.

a. Return to $k=3.2$ in your spreadsheet. Set the first value of $x=0.30$ and record the last 10 values of x (i.e., $n=191$ to $n=200$). Repeat for a starting value of $x=0.29$. Is there a change in the final 10 values?

b. Now set $k=3.7$ and repeat part (a).

c. A hallmark of chaotic motion is extreme sensitivity to initial conditions. That is, if you have two identical systems that start off with slightly different initial values, in a chaotic situation the two systems will look less and less alike as time goes on, whereas in non-chaotic situations the two systems will always evolve in a very similar way. Based on your results in (a) and (b) above, which value of k produces chaotic motion?

d. Try to determine the range or ranges of values of k for which chaotic motion results.

Concluding remarks on the population model:

The single fixed point $x=1-1/k$ is stable from $k=1$ to $k=3$. At $k=3$, the single fixed point becomes unstable while the fixed point pair (that we solved for above) becomes stable. That is, eventually the value of x will settle down to an oscillation between two fixed values. At $k=3.449490\dots$, this fixed point pair becomes unstable and a set of four fixed points defines a new, stable oscillation of x among 4 different values. This process is known as period doubling. There is another period doubling at $k=3.544090$ (to 8 points), and so on. The doublings occur at closer and closer spacings until at $k=3.569945\dots$ the chaotic region is reached. The system is not chaotic from here until $k=4$, however. There are regions within this range where other stable cycles are found, including cycles with odd numbers of fixed points. For instance, when $k=3.828435$, there is an orbit with three fixed points. These stable orbits are sometimes referred to as “attractors” because, due to their stability, the system evolves in such a way as to approach the orbit.

The Duffing Oscillator

An essential feature of chaotic systems is a nonlinear aspect. Thus, while the damped, driven harmonic oscillator will not produce chaotic motion, if the oscillator is not harmonic (that is, if the restoring force is not simply proportional to the displacement but instead is a more complicated function) there is the possibility that the resulting motion will be chaotic. The Duffing oscillator is an example of such a system. Look up the following paper:

J. E. Berger and G. Nunes, Jr. “A mechanical Duffing oscillator for the undergraduate laboratory.” *American Journal of Physics* volume 65 Number 9 pages 841-846 (1997). If you access this through the EMU library portal, you should be able to download a pdf of the entire paper free.

This paper describes a way of constructing a Duffing oscillator.

Problems:

7. In this problem, you will set up the equation to be analyzed.

a. Starting from the equation of motion,

$$ml^2\ddot{\theta} = mgl\sin\theta - \alpha\theta - k\dot{\theta} + F_o\cos\omega t$$

verify the authors' claim that this can be rewritten as

$$\ddot{\theta} \approx -a\theta^3 + b\theta - c\dot{\theta} + f\cos\omega t \quad .$$

Verify the authors' expressions for a, b, and c, and derive an expression for f.

b. Define a dimensionless time variable, $\tau = \frac{\omega t}{2\pi}$. With this

definition, the period of the driving force (as measured by the new time variable τ) is 1. Show that the differential equation can be rewritten as

$$\frac{d^2\theta}{d\tau^2} \approx -A\theta^3 + B\theta - C\frac{d\theta}{d\tau} + F\cos 2\pi\tau$$

and find expressions for A, B, C, and F in terms of a, b, c, and f. Use this form of the equation for all subsequent problems, unless otherwise noted. Also, assume in all subsequent problems that A, B, C, and F are all nonnegative and time independent.

8. Exploring the undriven case.

a. If there is no damping and no driving force, then $C=F=0$. Under these conditions, it is convenient to write

$$\frac{d^2\theta}{d\tau^2} = \frac{d}{d\theta} \left[\frac{1}{2} \left(\frac{d\theta}{d\tau} \right)^2 \right]$$

Verify the above expression, then insert it into the differential equation (with $C=F=0$). The equation is now easily integrated once with respect to θ . Do so, making sure to include an integration constant since you will be computing an indefinite integral. Discuss the resulting equation, in particular the way in which it is analogous to the conservation of mechanical energy. The equation you obtain will not have energy units but can nevertheless be written in the form $E_{\text{eff}} = T_{\text{eff}} + V_{\text{eff}}(\theta)$ where E_{eff} is a constant of motion, T_{eff} is proportional to the square of the angular velocity, and V_{eff} is a function depending on angular displacement only.

b. Using standard calculus techniques, show that V_{eff} has one local maximum and two local minima. Determine the values of θ and V_{eff} at each of those three points. Use this information to sketch $V_{\text{eff}}(\theta)$.

- c. Find the frequency, ω_0 , of small amplitude oscillations about either of the local minima.
- d. Discuss qualitatively the nature of larger amplitude oscillations and how the motion will change if C is no longer zero.

9. Finding the right solver: Refer back to Module C2 and write an M-file to solve the differential equation described in problem 7b, under the conditions that $A=3$, $B=1$, $C=0$, and $F=0$ (i.e., no driving force and no damping).

a. Using your results from Problem 8, determine the location of the two local minima for this case and predict a numerical value for the period. Now let MATLAB integrate this equation up until $t=100$, taking your initial position to be 0.05 to the left of the *minimum* on the positive θ axis and your initial velocity to be zero. Plot the result. Using your data or the plot, determine the period for this motion and compare to your prediction. For this part, submit a copy of your m-file, your predicted period, the measured period, and your plot.

b. Now have MATLAB solve the same problem but take the initial value of θ to be +0.05 (i.e., close to the unstable equilibrium point rather than close to a stable equilibrium point). When you plot the result, you will likely observe that the oscillator has managed to get over to the left hand potential well. For this part, submit a copy of your plot and an argument for why the observed result must be physically incorrect.

c. In addition to the ode45 solver that you presumably used in parts a and b, MATLAB has several other solvers. Fortunately, all of these solvers use the same syntax in their argument list, so we can change the solver just by changing “ode45” to another name. Here is a complete list of solvers:

ode45
ode23
ode113
ode15s
ode23s
ode23t
ode23tb

Repeat parts a and b using ode23tb. Submit plots of theta versus time and comment on which solver seems to work better. [Note, all of the solvers can be modified using an “option” command. Don’t

do that for this problem, but I just wanted to let you know that there are other ways to improve performance.]

d. We can also test the quality of the solver by checking to see if mechanical energy is conserved (it should be when $C=F=0$). In this case, the effective mechanical energy is

$$E_{\text{eff}} = \frac{1}{2} \left(\frac{d\theta}{d\tau} \right)^2 + \frac{A}{4} \theta^4 - \frac{B}{2} \theta^2$$

Have MATLAB calculate this energy as a function of time for both ODE solvers. This is easily done after running the ODE solver by adding a line of code $E=...$, remembering to perform all of the array calculations on an element by element basis. That is, if your angle variable is stored in an array named θ , you will have terms like “ $\theta.^2$ ”, not “ θ^2 ”. Note: I found it easier to spread the plots out along the y axis by plotting $1000 * E_{\text{eff}}$. For this part, submit plots of the effective energy versus time for both `ode45` and `ode23tb`, and for both initial conditions you examined above. There will be 4 plots altogether. Make sure they are clearly labeled. Discuss these results and come to a decision about which solver is likely to be more reliable. Neither one is perfect, but then again, there is no such thing as an exact numerical solver.

10. One lesson I hope you learned from the previous problem: when solving an equation numerically, first check to make sure the solutions conform to known results before you start exploring unknown regimes. While there is no guarantee a technique that correctly reproduces known results will continue to work correctly when you enter the unknown regime, you certainly improve your odds by checking the known first. In this problem, you will venture off into a bit less well-known territory by adding damping to your system.

a. Taking $\theta_0=0.050$, rerun the `ode23tb` solver (keeping $A=3$, $B=1$, $C=0$, and $F=0$). Rather than plotting the position and velocity as a function of time, plot the velocity as a function of position. [I stored my angles and angular velocities in a matrix called y , so my plot line read: `>> plot(y(:,1),y(:,2))`]. This is known as a phase plot. For true periodic motion, this plot should trace over itself each cycle. Due to the numerical approximations here, you won't see the plot exactly trace over itself, but it will come pretty close. Submit a copy of the plot and a discussion of its features (why it is shaped the way it is, what symmetries it has and why, etc.).

- b. Repeat part (a) but now take $C=0.075$. You should notice decaying oscillations.
- c. Repeat part (b), but now take the initial angle to be 1.0 instead of 0.05. Why does the phase plot look so much different this time?

11. Seeking chaos: Now let's add the driving force in and see if we can find a chaotic regime.

a. Continue using $A=3$, $B=1$, $C=0.075$, and add to this $F=5.0$. Using `ode23tb`, integrate the solution out to $t=40$, taking as your initial displacement 1.0 and your initial velocity 0. Does this look like a periodic solution? Make your best guess based on what you see at this point (not based on results from later parts). Submit your plot of position versus time and some discussion.

b. Now extend the integration time out to $t=400$. Does this change your conclusion? You may find it useful to plot data towards the end of your run. I found commands like `plot(t(11900:12400),y(11900:12400,1))`

useful for plotting just some of the stored values. The size of the t and y arrays vary each time you use the solver. You can monitor the size in the workspace window. Alternatively, to find how many entries there are in your t array, type in "`size(t)`". For this part submit a discussion and one plot that supports your new conclusion.

c. Vary the drive amplitude F to see what other forms of long term behavior you can find. I was unable to find any chaos myself, but I did come across something beyond basic oscillatory motion. Submit a discussion of your findings, and plots if appropriate.

d. Note that the natural frequency near one of the local minima is $\sqrt{2B}$, corresponding to a period of $\frac{2\pi}{\sqrt{2B}} = \frac{2\pi}{\sqrt{2 \cdot 1}} = 4.44$, well off from

our drive period, which has been set to 1 in dimensionless units.

Perhaps if we drive the system closer to resonance, something more interesting will happen. With this in mind, let's set $B=15$ (so that the natural period is 1.15). We'll increase A to 45 to keep the local minimum in the same spot as before. In summary, use $A=45$, $B=15$, $C=0.075$, $F=1.0$ and integrate out to $t=400$. Discuss the nature of the solution at long times.

e. Repeat (d) but take $F=3.0$ and then $F=20$. Why do you suppose the result for $F=3.0$ is so much different?

12. Studying chaos: From the previous problem, you may have deduced that when $A=45$, $B=15$, and $C=0.075$, we might observe chaos when $F=3.0$ but we definitely do not see it when $F=1.0$. For $F=20.0$, we probably did not see chaos.

a. Produce phase space plots (as in problem 10) for the three different values of F . To eliminate the short term, transient behavior, only plot data for (approximately) $200 < t < 400$. Submit the plots and a discussion.

b. Now check for sensitivity to initial conditions. For $F=1.0$, run two integrations. The first will have the form

```
[t,y]=ode23tb(@pendulum,[0,10],[1.0;0]);
```

while the second will use slightly different initial conditions:

```
[tnew,ynew]=ode23tb(@pendulum,[0,10],[1.01;0]);
```

The positions as a function of time can be plotted on a single page using the following series of commands:

```
>> plot(t,y(:,1))
```

```
>> hold on
```

```
>> plot(tnew,ynew(:,1))
```

```
>> hold off
```

The “hold on” command tells MATLAB to hold the current plot while adding new data to it. “Hold off” means that next time you issue a plot command, previously plotted data will be erased first. Submit your plot for this part.

c. Repeat part (b), now taking $F=3.0$. Submit a plots and a discussion of the difference between the two results. Does your conclusion support the hypothesis that $F=1.0$ is not chaotic while $F=3.0$ is?

13. Exploration: Vary A , B , C , and F to identify another chaotic regime (not close to the one you just found). Submit sufficient plots and discussions to justify your conclusion.

MODULE NED1

Name:

Term:

| Problem | Max | 1 st | 2 nd | Final |
|---------|-----|-----------------|-----------------|-------|
| 1 | 5 | | | |
| 2 | 5 | | | |
| 3 | 5 | | | |
| 4 | 10 | | | |
| 5 | 5 | | | |
| 6 | 5 | | | |
| 7 | 5 | | | |
| 8 | 10 | | | |
| 9 | 10 | | | |
| 10 | 5 | | | |
| 11 | 10 | | | |
| 12 | 5 | | | |
| 13 | 10 | | | |
| TOTAL | 90 | | | |