Module M2

Lagrangian Mechanics and Oscillations

Prerequisite: Module C1

This module requires you to read a textbook such as Fowles and Cassiday on material relevant to the following topics.

Topics:

Hamilton's Variational Principle
Lagrange's Equations and Generalized Coordinates
Generalized Momenta
Lagrange Multipliers and Forces of Constraint
Generalized Forces
Hamilton's Equations

Equilibrium and Stability
Oscillators With One Degree of Freedom
Coupled Oscillators
Vibrating Systems
Continuum Limit

References:

Fowles and Cassiday *Analytical Mechanics* 7th ed (Thomson/Brooks/Cole 2005) Chapter 10, Chapter 11

Thornton and Marion *Classical Dynamics of Particles and Systems* 5th ed (Thomson/Brooks/Cole 2004) Chapter 7 (sections 1-11), Chapter 12, Chapter 13 (sections 1-4)

Assignment:

Note that many of these problems are greatly influenced by (and in some cases virtually identical to) Fowles and Cassiday

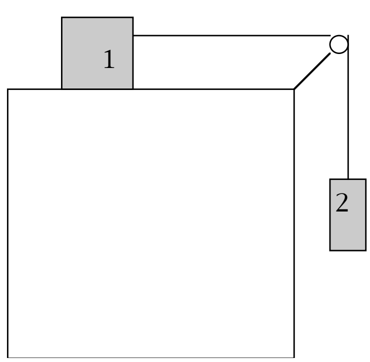
In the problems 1-6, you must use the Lagrangian approach.

1. An object of mass m is suspended from an ideal massless vertical spring of spring constant k. Assuming there is no friction in the system, derive the differential equation of motion (i.e., an expression for the acceleration), assuming the object moves only in the vertical direction.

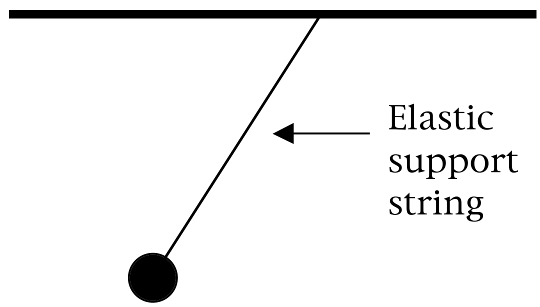
2. A solid cylinder rolls without slipping down a ramp inclined at an angle θ with respect to the horizontal. Determine its acceleration.

3. In a situation similar to problem 2, now assume the ramp sits on a frictionless horizontal surface, so that it is free to move also. Take M to be the mass of the ramp and m to be the mass of the cylinder. Find the acceleration of the ramp.

4. In the figure below, the table is frictionless, block 1 has mass m_1, block 2 has mass m_2, and the rope is uniform with total mass m_3 and length ℓ. The mass of the pulley is negligible. Find an expression for the acceleration of the blocks.



5. Consider a simple pendulum confined to move in a plane, but with the support rod replaced by an elastic string of stiffness k (equivalent to a spring constant) and unstretched length l_o. Ignore the mass of the string and take the mass of the bob to be m. Find the differential equations of motion for this system using polar coordinates. Assume that, due to the weight of the pendulum bob, the string is always stretched past its unstretched length.



6. Expressing the square of the speed of a particle in spherical coordinates as $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + (r\sin\theta)^2\dot{\phi}^2$, find the differential equations of motion for a particle of mass m in a central potential V(r). Re-express these equations in terms of the radial force. Note that a central potential has no θ or ϕ dependence.

7. A track of mass M has a surface representing one quarter of a circle of radius b. The track sits on a frictionless, horizontal surface. A small block of mass m slides without friction down the track, after having been released from rest when $\theta=0$. Using the Lagrange multiplier method and the coordinate system shown,

a. show that the motion is governed by the three equations,

$$M\dot{x}_T + m\dot{x}_T + mb\dot{\theta}\sin\theta = const \qquad (1')$$

$$g\cos\theta - b\ddot{\theta} - \ddot{x}_T\sin\theta = 0 \qquad (2')$$

$$b\dot{\theta}^2 + g\sin\theta + \ddot{x}_T\cos\theta + \frac{\lambda}{m} = 0 \qquad (3')$$

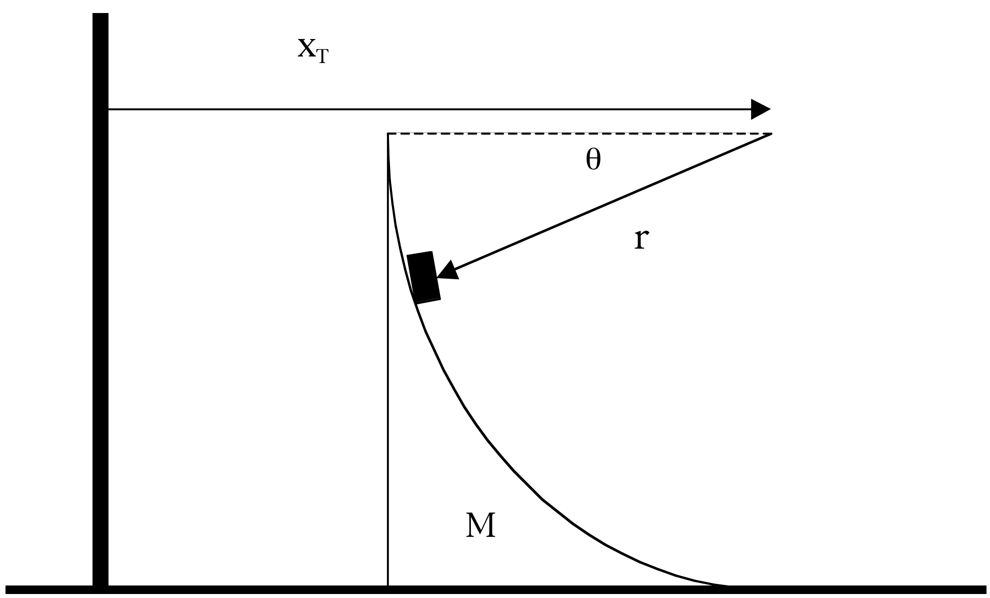
b. Letting $\gamma=m/(M+m)$, show that 2' and 3' can be rewritten as

$$g\cos\theta - b\ddot{\theta} + \gamma b\frac{d}{dt}\left(\dot{\theta}\sin\theta\right)\sin\theta = 0 \qquad (2'')$$

$$b\dot{\theta}^2 + g\sin\theta - \gamma b\frac{d}{dt}\left(\dot{\theta}\sin\theta\right)\cos\theta + \frac{\lambda}{m} = 0 \qquad (3'')$$

c. From these equations, show that the constraint force on the block is given by (I think!):

$$mg\left[-\sin\theta + \frac{-2\sin\theta + 2\gamma\sin^3\theta + 3\gamma\cos^2\theta\sin\theta - \gamma^2\cos^2\theta\sin^3\theta}{\left(1-\gamma\sin^2\theta\right)^2}\right]$$



8. For each of the following potentials, determine (i) the equilibrium position(s) (ii) the conditions under which the equilibrium is stable, and (iii) the oscillation frequency, ω, if the equilibrium is stable. Take the mass of the particle in the potential to be m.

a. $V(x) = \frac{1}{2}kx^2 + k'x^4 \quad \textit{where } k > 0$

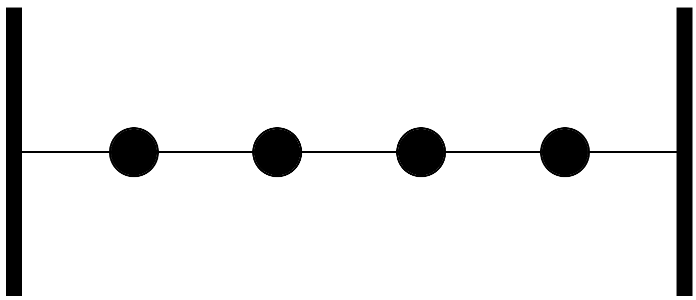
b. $V(r) = V_o\left[\left(\frac{a}{r}\right)^{12} - \left(\frac{a}{r}\right)^{6}\right] \quad \textit{where } V_o > 0, a > 0, r > 0$

c. $V(x) = Bxe^{-x/a} \quad \textit{where } a > 0$

9. A double pendulum is constructed by attaching a string of length L to a pivot point. At the base of the string is an object of mass m_1(you can treat it as a point mass). Tied to the bottom of this object is a second string of length L, and at the base of that is a second point-like mass, m_2. Let $\alpha=m_2/m_1$.

a. Derive a general expression for the normal mode frequencies for this system, expressing your final answer in terms of L and α. You may assume small amplitude oscillations.
b. Simplify this expression in the special case that $\alpha=1$.
c. Simplify this expression in the special case that $\alpha<<1$ (but not $\alpha=0$!).
d. Simplify this expression in the special case that $\alpha>>1$ (but not infinitely large!).
e. Create a computer generated plot of the normal mode frequencies as a function of α.

10. An array of 4 identical particles of mass m are joined by identical springs of stiffness k. Both ends of this array are attached to fixed bracket with an additional spring.



Determine the normal mode frequencies for longitudinal oscillations. Is there any mode in which all of the particles are always moving in the same direction, that is, where they are all moving to the left at one time and all moving to the right at another, but never some moving to the left and some moving to the right?

11. Find a derivation of the equation for waves on a uniform string, and write it out in detail, carefully justifying each step. The point of this exercise to make sure you understand the origin of this equation. The equation you derive should have the form

$$\frac{\partial^2 q}{\partial t^2} = v^2 \frac{\partial^2 q}{\partial x^2}$$

where q(x,t) represents the displacement of part of the string (at position x) from its equilibrium position.

MODULE M2

Name: Term:

Problem	Max	1st	2nd	Final
1	5			
2	5			
3	5			
4	5			
5	5			
6	5			
7	10			
8	5			
9	10			
10	5			
11	5			
TOTAL	65			