Module E1

Boundary Value Problems in Electrostatics and Special Functions

Prerequisite: Module C1

This module requires you to learn material relevant to the following topics from a textbook such as Griffiths. You may find it useful to skim through the first few problems and then begin to read your textbook so that you know what material to look for.

Topics:

Laplace's Equation
Uniqueness Theorems
Method of Images
Separation of Variables in Cartesian and Spherical Coordinates
Multipole Expansion
Legendre Polynomials

References:

David Griffiths *Introduction to Electrodynamics* 2nd ed (Prentice Hall 1989) Chapter 3

Paul Lorrain and Dale Corson *Fundamentals of Electromagnetic Phenomena*

Assignment:

Note that some of these problems are greatly influenced by (and in some cases virtually identical to) Griffiths

1. The first uniqueness theorem for Laplace's equation states that in a given region, there exists just one solution to Laplace's equation provided the value of the potential, V, is known everywhere on the boundary of the region. Prove this using the following approach:

- Let $V(\mathbf{r})$ represent a solution to Laplace's equation in the given region subject to the specified boundary conditions.
- Define $V_{new}(\mathbf{r})=V(\mathbf{r})+f(\mathbf{r})$ where f is an unknown function such that V_{new} satisfies Laplace's equation in the same region with the same boundary conditions.
- Show that under these conditions, f must be zero everywhere in this region and hence $V=V_{new}$.

2. A thin "wire" of plastic is infinitely long and has uniform charge per unit length equal to λ. It sits a distance d above an infinite grounded conducting sheet in the xy plane. Take the wire to be parallel to the y axis.

a. Determine the potential in the space above the conducting sheet.

b. Determine the induced charge density on the conducting sheet.

c. Plot the electric potential along the line given by x=d, y=0, z>0. On the same plot, show the potential in the absence of the conducting sheet. In the latter case you will need to choose a reference point. Take V=0 at the point x=d, y=0, z=0 (this will match the boundary condition in the first potential).

3. An infinite, grounded, conducting plane lies in the z=0 plane.
a. A particle of mass m and charge q is held at the point $(0,0,z_o)$. What electric field does it experience due to the induced charge on the plane?

b. Show that if the particle is released from rest at the point (0,0,D), the time it would take for it to strike the sheet is given by

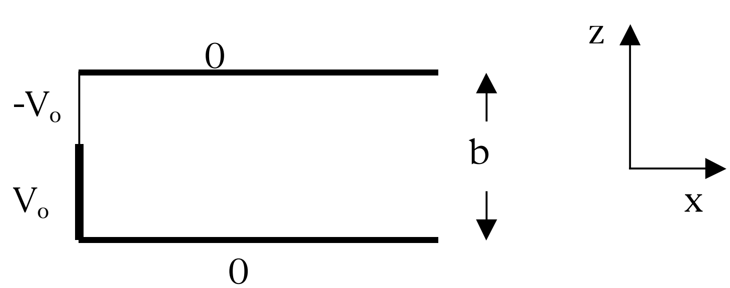
$$T = \sqrt{\frac{8\pi\varepsilon_o m D}{q^2}} \int_0^D \sqrt{\frac{z/D}{1 - z/D}}\, dz \quad .$$

c. Using the trig substitution $z/D = \sin^2\theta$, complete this integral.

4. An infinite slot is formed by 4 infinite conducting sheets, separated by insulating strips wherever they might otherwise touch:

- Sheet A is semi-infinite, located in the x>0 portion of the z=0 plane and is kept at a potential of 0.
- Sheet B is semi-infinite, located in the x>0 portion of the z=b plane and is kept at a potential of 0.
- Sheet C is an infinitely long strip in the yz plane, with one boundary along the x=0, z=0 line and the other boundary along the x=0, z=b/2 line. It is maintained at a potential of V_o.
- Sheet D is an infinitely long strip in the yz plane with one boundary along the x=0, z=b/2 line and the other along the x=0,z=b line. It is maintained at a potential of $-V_o$.

Looking down the y axis we would see:



Calculate the potential inside the slot.

5. A cubical box is made of six metal squares (each edge of length b) that are attached to each other by insulating strips, so that each square can be maintained at a different potential. Take the origin of the cube to be at one corner and each of its edges to be parallel to an axis. The cube is oriented to that it is in the positive octant. The cube face lying in the z=0 plane is maintained at a potential of $-V_o$, the face lying in the z=b plane is maintained at a potential of $+V_o$, while the other four faces are maintained at a potential of 0. Find the potential as a function of position inside the cube.

6. Legendre Polynomials

a. Show that $\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)(\sin\theta)\Theta$ can be rewritten as $\frac{d}{dx}\left[\left(x^2-1\right)\frac{dP}{dx}\right] = l(l+1)P$ provided x=cosθ and P(x)=Θ(θ).

b. The first four Legendre polynomials, $P_l(x)$ can be rewritten as

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}\left(3x^2-1\right)$$

$$P_3(x) = \frac{1}{2}\left(5x^3-3x\right)$$

Show by direct substitution that these polynomials satisfy the differential equation for P(x).

c. One way of obtaining the Legendre Polynomials is through the Rodrigues formula:

$$P_l(x) = \frac{1}{2^l(l!)}\left(\frac{d}{dx}\right)^l\left(x^2-1\right)^l$$

Verify each of the above four Legendre Polynomials can be found this way.

d. Using the Rodrigues formula show that the highest power term in any Legendre polynomial has the form $\frac{(2l)!}{2^l(l!)^2}x^l$

e. The orthogonality relationship for Legendre polynomials can be given by one of two expressions:

$$\int_{-1}^{1} P_l(x)P_m(x)\,dx = 0 \quad if\ l \neq m$$

$$\int_{0}^{\pi} P_l(\cos\theta)P_m(\cos\theta)\sin\theta\,d\theta = 0 \quad if\ l \neq m$$

Verify that these two expressions are equivalent.

f. Using either one of the above integral expressions, verify that P_1 is orthogonal to P_2, P_1 is orthogonal to P_3, and P_2 is orthogonal to P_3.

7. Another approach to finding the Legendre Polynomials.

a. Starting from $\frac{d}{dx}\left[\left(x^2-1\right)\frac{dP}{dx}\right]=l(l+1)P$, assume the solutions can be written in the form $P(x)=\sum_{n=0}^{\infty} b_n x^n$. Show that

$$b_{n+2}=\frac{n(n+1)-l(l+1)}{(n+2)(n+1)}b_n$$

b. What does this expression tell us about the degree of the polynomial associated with P_l?

c. The recursion relation determines the polynomials only up to a constant. Show that it correctly predicts b_3/b_1 for P_3 and b_2/b_0 for P_2, using for comparison the polynomials given in the previous problem.

8. Consider a hollow sphere of radius b centered at the origin. With the aid of batteries and appropriate insulating dividers, the potential on the surface of the sphere is given by

$$V(\theta,\varphi)=V_o \text{ when } 0<\theta<\frac{\pi}{4}$$

$$V(\theta,\varphi)=0 \text{ when } \frac{\pi}{4}<\theta<\frac{3\pi}{4}$$

$$V(\theta,\varphi)=-V_o \text{ when } \frac{3\pi}{4}<\theta<\pi$$

a. Find an expression for the electric potential outside this sphere using an expansion based on Legendre polynomials. Keep the first 2 nonzero terms in the expansion.

b. How large does r need to be in order to be sure that the second term in the expansion is at least 100 times smaller than the first term for all values of θ?

9. A linear charge distribution of the form

$$\lambda = \beta\left(z + \frac{b}{2}\right) \quad -\frac{b}{2} \le z \le \frac{b}{2}$$

$$= 0 \quad |z| > \frac{b}{2}$$

lies on the z axis.

a. Using the multipole expansion, calculate the first two terms of the electric potential for this distribution using spherical coordinates.

b. Calculate the electric field of this charge distribution in spherical coordinates, again using the first two terms in the multipole expansion.

MODULE E1

Name: Term:

Problem	Max	1st	2nd	Final
1	5			
2	10			
3	5			
4	5			
5	5			
6	15			
7	5			
8	5			
9	5			
TOTAL	60			