Experiment 4: Harmonic Motion Analysis

Background

In this lab you will investigate the influence of damping on a driven harmonic oscillator and study resonant conditions.

The following theoretical development is based on Analytical Mechanics by Fowles (4th edition). A mass m attached to a spring with spring constant k and experiencing no dissipative forces is governed by the differential equation

$$m\frac{d^2x}{dt^2} + kx = 0 \qquad (1)$$

where x represents the position as a function of time t. The solution to this equation may be expressed as

$$x = A_1 \cos(\omega_o t + \theta_o) \qquad (2)$$

where A_1 is the amplitude of the oscillation, θ_o is a phase factor determined by the initial conditions and the *natural frequency* ω_o is given by

$$\omega_o = \sqrt{\frac{k}{m}} \quad . \qquad (3)$$

In the case that linear damping is present, there is an additional force term in the governing differential equation:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \quad . \qquad (4)$$

In this equation, c is known as the damping constant. There are three classes of solutions to this equation. Of particular interest to us is the underdamped case, whose solution takes the form

$$x = A_1 e^{-\gamma t} \cos(\omega_d t + \theta_o) \qquad (5)$$

where we will refer to γ as the damping factor (to distinguish it from the damping constant) and it takes on the value

$$\gamma = \frac{c}{2m} \quad . \qquad (6)$$

Even though the motion strictly speaking is no longer periodic (the amplitude is decaying), ω_d is still interpreted as an oscillation frequency. It takes on the value

$$\omega_d = \sqrt{\omega_o^2 - \gamma^2} \quad . \qquad (7)$$

Note that if we were to measure the amplitude of a decaying oscillation on 2 occasions, separated by n cycles, the ratio of those amplitudes, according to equation (5), would be

$$\frac{A_o}{A_n} = e^{\gamma n T_d} \qquad (8)$$

where

$$T_d = \frac{2\pi}{\omega_d} \quad . \qquad (9)$$

Finally, suppose we apply a periodic driving force to the oscillator. We can do this through a periodic displacement of one end of the spring. A periodic displacement of the form $A_{drive}\cos(\omega t)$ will produce a force $kA_{drive}\cos(\omega t)$, resulting in an equation of motion

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = kA_{drive}\cos(\omega t) \quad . \qquad (10)$$

The solution to this equation contains a decaying portion which vanishes once the oscillation reaches steady state. The steady state portion is

$$x = A\cos(\omega t + \phi) \qquad (11)$$

where

$$A = \frac{kA_{drive}}{\sqrt{\left(k - m\omega^2\right)^2 + c^2\omega^2}} \qquad (12)$$

and

$$\tan\phi = \frac{c\omega}{k - m\omega^2} \quad . \qquad (13)$$

When the drive frequency ω is adjusted so that the amplitude is maximized, we say that resonance has been achieved. The resonant frequency is given by

$$\omega_r = \sqrt{\omega_o^2 - 2\gamma^2} \tag{14}$$

It is left as an exercise to the reader (and you will need to show this in your introduction) that

$$\omega = \omega_o \Rightarrow \phi = 90^o \tag{15}$$

and at resonance for a weakly damped oscillator

$$A \approx \frac{\omega A_{drive}}{c/m} = \frac{\omega A_{drive}}{2\gamma} \quad . \tag{16}$$

Procedure:

Apparatus: Pasco Driven Harmonic Motion Analyzer

1. The apparatus should already be set up as indicated in Figure 1 of the equipment manual in the lab, using a red spring. D0 NOT TAKE THE HANGING BARS OFF THE SPRING. IT IS NOT NECESSARY TO MEASURE THEIR MASS. Rotate the screws holding the two damping magnets so that they are as far from the damping rod as possible.

2. Carefully align the apparatus as indicated on page 4 of the apparatus manual (follow both steps one and two). If this is not done correctly, your subsequent measurements will not be reliable.

3. You will now determine the resonant frequency three different ways:

A. With the drive motor off, set the function switch period. Pull the rod down a centimeter or two and release. Note the value of the period.

B. The drive amplitude A_{drive} has been set. Do not adjust this setting. (see Figures 9 and 11 in the equipment manual). Turn on the drive motor and set the function switch to amplitude. Vary the drive frequency until the amplitude is a maximum. The resonant frequency can now be read by turning the function switch back to frequency. (If the oscillations are too violent, you can move the magnets in slightly to reduce the oscillation amplitude.)

C. Vary the frequency until the phase indicator reads a lag of 90^{o}.

In all three of these methods you should obtain *approximately* the same answer. The first frequency you measured was $f_d = \omega_d/2\pi$, and the third was $f_o=\omega_o/2\pi$. The second was the true resonant frequency. Be sure to explain in your report why these are in fact the frequencies you measured.

4. In the following steps, you will measure the damping factor γ as a function of magnet-rod separation. First, repeat your alignment process of step 2. Then, with the drive motor off, check that the magnets are retracted as far as possible and set the function switch to amplitude.

A. Pull the rod down as far as practical without going past the point at which measurements can be made.

B. Release the rod and note the first reliable amplitude measurement.

C. Record the amplitude exactly three cycles later.

D. Repeat steps A-C but wait five cycles instead of three.

E. Repeat steps A-C, waiting ten cycles this time.

5. With the help of equation (8), you now have enough information to determine the damping factor. The apparatus has measured the time dependent amplitude to the sine function (actually twice that value since it reads peak to peak). You have two separate readings separated by a known time (n cycles where you know the time it takes to complete each cycle). Note that each of the three trials above should give you a value of the damping factor and ideally, they should all three be the same.

6. Measure the distance between the magnets using calipers. Then, move *each* magnet in by turning it one complete revolution. A small piece of tape should be on the magnet to help guide you. Turning one magnet one complete revolution moves it in 0.130 +/- 0.005 cm. Now repeat steps A-E. Continue moving the magnets in one complete turn each, making a complete set of measurements each time, until you have data for 5 different magnet separations. This will give you γ as a function of magnet-rod distance, which you should *plot* in your report. In your report you should discuss qualitatively the physics behind this damping mechanism and why you would expect the plot of γ vs. magnet distance to look as it does.

7. In this final step, you will measure the distance dependence of the damping constant by looking at driven oscillations. The drive amplitude, A_{drive}, has been set to 1.15 +/- 0.10 mm. Start with the magnets all the way out. Turn the drive motor on and make sure you are still in resonance. Measure the amplitude of oscillation of the hanging mass using the amplitude setting on the apparatus. (Note this measures *peak to peak* amplitude!) Repeat this process for all of the magnet separations used in step 4 above. This amplitude, along with equation (15) above, will allow you to solve for γ. Compare these results graphically with those of steps 4-6 by displaying this data on the *same plot*.

What is due: You should write up all of the sections you wrote for the previous report and add a Conclusion. It is important in your Conclusion that you make a connection with the objectives you stated in your Introduction. This will also serve as a reminder to make explicit in your Introduction what your objectives are! Your Introduction might contain a statement towards the end such as, "The objective is to verify equations 7 and 9 described above." In your Conclusion, you would discuss whether your data is in fact consistent with equations 7 and 9. In deciding whether this is the case, it does not matter if theory is "close" to experimental results. "Close" is a subjective judgment. What matters is whether theory and experiment agree to within your calculated uncertainties.

- If the agreement is within calculated uncertainties, then you can conclude that the results of this experiment are consistent with the theory. You can go on to suggest ways in which the experiment might be modified to further reduce the uncertainties and thus provide a more rigorous test of the theory.

- If the agreement is not within calculated uncertainties, then *do not* conclude that the theory has been verified or nearly verified. It is tempting to state a positive result, but sometimes experiments do not work out the way we want them to. You need to honestly conclude there is something about this experiment that you do not yet understand. Perhaps you underestimated the uncertainties. Perhaps there is a source of systematic error that has been neglected. Perhaps there were approximations made in the theoretical development that do not in fact apply to this experiment. You should identify which of these explanations are the most likely to be relevant and provide some, crude quantitative estimate justifying your conclusion. For instance, if you think friction was the culprit, you might say something like "We noted that the rotational speed decreased by about 1% every revolution and hence friction might account for a 1% discrepancy between theory and experiment."

Finally, pay careful attention to notes I made on your previous write-ups. If you have any questions about my comments, please ask. I do not intend this to be a game of "Try And Guess What The Instructor Is Really Looking For." My goal is to spell out exactly what components are necessary to write a good report for this course.