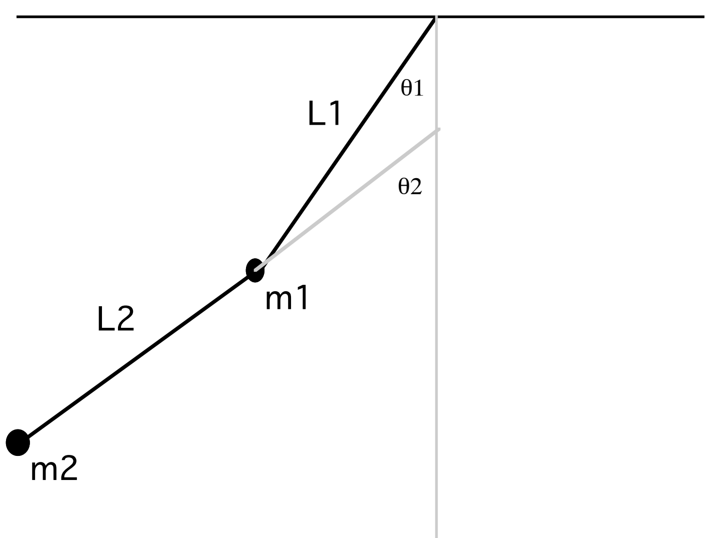
Experiment 3: Multicomponent Oscillators

Background

In this lab, you will observe normal modes and beating in oscillators with two degrees of freedom.

For the double pendulum shown below the upper mass



has speed

$$v_1 = L_1 \dot{\theta}_1 \quad (1)$$

while the lower has

$$\begin{aligned} v_2 &\approx v_{2/1} + v_1 \\ &\approx L_2 \dot{\theta}_2 + L_1 \dot{\theta}_1 \end{aligned} \quad (2)$$

Note the approximation in v_2 comes in that the velocity vectors are not exactly parallel in general, so the sum should be a vector sum instead of a scalar sum. This is the first in a series of small amplitude approximations.

From (1) and (2), the total kinetic energy of this system is found to be

$$KE \approx \frac{1}{2} m_1 \left(L_1 \dot{\theta}_1 \right)^2 + \frac{1}{2} m_2 \left(L_2 \dot{\theta}_2 + L_1 \dot{\theta}_1 \right)^2 \quad (3)$$

Taking the zero point of the potential energy to occur when both masses hang straight down, then when they are raised,

$$PE = m_1 g\left(L_1 - L_1\cos\theta_1\right) + m_2 g\left[\left(L_1 - L_1\cos\theta_1\right) + \left(L_2 - L_2\cos\theta_2\right)\right] \quad (4)$$

If θ, as measured in radians, is sufficiently small, then one can approximate $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ so that

$$PE \approx \frac{1}{2} m_1 g L_1 \theta_1^2 + \frac{1}{2} m_2 g\left(L_1\theta_1^2 + L_2\theta_2^2\right) \qquad (5)$$

Equations (4) and (5) determine the energetics of the system. From them, the equations of motion can be derived. This is most conveniently done using Lagrange's Equations (see Analytic Mechanics by Fowles and Cassiday). While it is not expected that most students taking this course will be familiar with the Lagrange technique, it is nevertheless useful to note that the Lagrange approach allows for the derivation of the equations of motion (6) much more readily than F=ma. The result is the following coupled set of equations:

$$\begin{aligned} &\left(m_1 + m_2\right)L_1^2\ddot{\theta}_1 + m_2 L_1 L_2\ddot{\theta}_2 + \left(m_1 + m_2\right) g L_1 \theta_1 = 0 \\ &m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \ddot{\theta}_1 + m_2 g L_2 \theta_2 = 0 \end{aligned} \qquad (6)$$

Understanding the following definition is essential to carrying out the procedure: A *normal mode* of the system is defined as a state of motion in which all the elements oscillate at the *same* frequency. Such a state could have the mathematical form

$$\begin{aligned} \theta_1 &= \theta_{1o}\cos\omega t \\ \theta_2 &= \theta_{2o}\cos\omega t \end{aligned} \qquad (7)$$

Trying this form in (6) gives

$$\begin{aligned} &-\left(m_1 + m_2\right)L_1^2\omega^2\theta_{1o} + \left(m_1 + m_2\right) g L_1 \theta_{1o} - m_2 L_1 L_2 \omega^2 \theta_{2o} = 0 \\ &-m_2 L_1 L_2 \omega^2 \theta_{1o} - m_2 L_2^2 \omega^2 \theta_{2o} + m_2 g L_2 \theta_{2o} = 0 \end{aligned} \qquad (8)$$

which has a solution provided the determinant of the coefficients vanishes:

$$\begin{vmatrix} -\left(m_1 + m_2\right)L_1^2\omega^2 + \left(m_1 + m_2\right) g L_1 & -m_2 L_1 L_2 \omega^2 \\ -m_2 L_1 L_2 \omega^2 & -m_2 L_2^2 \omega^2 + m_2 g L_2 \end{vmatrix} = 0 \qquad (9)$$

Special Case 1: $L_1 = L_2$. Then (9) implies

$$\omega^4 - 2\left(1 + \frac{m_2}{m_1}\right)\frac{g}{L}\omega^2 + \left(1 + \frac{m_2}{m_1}\right)\frac{g^2}{L^2} = 0 \qquad (10)$$

Special Case 2: $L_1 = L_2$ and $m_2 << m_1$. Then the solutions to (10) are

$$\omega_{\pm} \approx \sqrt{\frac{g}{L}}\left(1 \pm \frac{1}{2}\sqrt{\frac{m_2}{m_1}}\right) \qquad (11)$$

and the amplitude ratios are

$$\begin{aligned} \frac{\theta_{2o}}{\theta_{1o}} &\approx -\sqrt{\frac{m_1}{m_2}} \quad \text{if } \omega = \omega_+ \\ \frac{\theta_{2o}}{\theta_{1o}} &\approx +\sqrt{\frac{m_1}{m_2}} \quad \text{if } \omega = \omega_- \end{aligned} \qquad (12)$$

The solution corresponding to ω_+ is known as the out-of-phase mode, since the angular displacements are related to each other by a minus sign. When the lower mass is displaced to the left, the upper will be displaced to the right, and vice versa. This is the higher frequency mode. The solution corresponding to ω_- is known as the in-phase mode because the two masses are always on the same side of the vertical—they oscillate in phase. You will be studying these two modes in lab.

Two modes of vibration with frequencies very close to each other will cause a beating phenomenon in which energy is transferred back and forth between the two modes at a slow rate compared to the vibrational mode frequencies. This will be seen as a slow variation in the amplitude of oscillation superimposed on the rapid periodic motion of the masses. The frequency of this slow variation is the beat frequency. A similar phenomenon occurs with sound waves. See an introductory text such as Halliday, Resnick and Walker for further details.

The beat frequency is associated with two modes whose frequencies are f_1 and f_2 is given by

$$f_{beat} = f_1 - f_2. \qquad (13)$$

The beating phenomenon may also be observed when one frequency is close to an integral multiple of the other frequency.

Beating can be looked at as a resonance phenomenon. If all of the energy starts off in one mode of oscillation, that mode acts as an oscillating driving force for the other mode. If the second mode has a frequency nearly equal to the first mode (or related by an integral multiple), then the first mode will be driving the second in resonance, thus very effectively transferring its energy to the second mode. Once the energy transfer is complete, then the second mode begins to act as the driving force for the first mode. This process will be observed in both the double pendulum system and the vertical mass and spring system.

Procedure:

Apparatus: A ring stand with masses that can be suspended from a spring or from thread, a stopwatch, and miscellaneous springs, hanging masses and washers.

The apparatus is simple for this experiment, but producing good results takes patience! For most measurements I have been able to get agreement between theory and experiment to within a few percent.

1. Set up a single pendulum using a 100g mass and with a length as close to 50 cm as you can get. The length is properly measured from the pivot point to the center of mass of the pendulum bob. I construct this by cutting a piece of thread about 75 cm long and first tying one end to the hook on the 100g mass. I lay the thread out along a meter stick to see where the top of the loop at the pivot end needs to be. *All masses used in this experiment should be checked on the scale. Do not trust the labels. Come as close as you can to the desired masses, given what is available in the lab.*

2. As a preliminary check, measure the period of this simple pendulum. Displace the 100g mass slightly and release. Time the resulting period. To reduce error, time 5-10 consecutive oscillations and compute the average. Repeat this measurement several times. Compare your result to that for the simple pendulum, $T = 2\pi\sqrt{L/g}$.

3. Next set up a double pendulum by attaching a 2g mass to the bottom of the 100g mass. You will see a hook on the bottom of the 100g mass to which the thread can be tied. The length of the lower pendulum should also be as close to 50cm as you can make it, with the length once again being measured from the pivot point for the 2g mass to its center of mass. Search for the out-of-phase mode by displacing the lower (2 g) mass to the left approximately 7 cm and the upper (100 g) mass to the right approximately 1 cm (i.e., hardly at all). (Equation (12) in the Introduction will help you see why I chose 7:1 ratio.) Release. If you have found the mode, you should see the two masses swinging in opposite directions with roughly *constant* (though different) amplitudes. If their amplitudes vary over time, you are observing the beating mode, not the out-of-phase mode. When you have achieved the out-of-phase mode, measure the period as above. It should be 5%-10% different from the value you obtained in step 2, with the exact prediction determined by equation (11). In your report, you should compare your measured period to that which can be obtained from the theoretical prediction in equation (11).

4. Repeat step #3 for the in-phase mode, which is found by displacing the bottom mass 7 cm and the top mass 1 cm in the same direction. Again, if you see the oscillation amplitude varying over time, you have found the beating mode, not the in-phase mode. This mode will likely be harder to find than the out-of-phase mode. If you are having trouble getting the system into this mode, you will observe periods when the bottom mass is swinging with larger than average amplitude while the upper mass swings barely at all. During one of these times, try pushing very lightly on the upper mass in the same direction that the lower mass is traveling. Repeat as necessary until you do not observe any significant variation in the amplitude of oscillation for either mass. I find it easiest to locate this mode when I keep the amplitudes small.

5. Without displacing the top mass, displace the bottom 10 - 20 cm and release. You should observe that the bottom mass first swings with a large amplitude, then transfers most of its energy to the upper mass, causing it to swing. If you have set this up properly, the top mass will come to almost a dead stop momentarily. Energy eventually flows back to the upper mass from the lower mass, allowing it to swing again. Then energy flows out of the upper mass into the lower mass, causing the upper mass to come to a momentary standstill again. This is one complete cycle of the beating. Time it and compare to what theory would predict. It is probably easiest to time the beat period by focusing your attention on the top mass, starting when it is not oscillating at all (i.e., only the bottom mass is oscillating) and stopping when it returns to that state.

When you are finished with the double pendulum experiments, you will do yourself a big favor by performing the calculations on the spot. Sometimes students do not realize they have incorrectly identified the modes until after they leave they have disassembled their double pendulum, at which point it becomes difficult to retake data without redoing the entire experiment. Remember, careful measurements should yield results good to within a percent or two. Once you are confident in your numbers, cut all strings off of the masses. (You may wish to double check your length measurements before you do this!)

6. While in the previous part of the lab, the two degrees of freedom were the two masses, in this part you will use a single mass capable of oscillating in two ways--as a pendulum and as a vertical mass and spring system. Suspend a *red* spring from the horizontal bar.

7. Determine the spring constant by measuring the change in the length of the spring due to a 50 g mass. Repeat with other masses available in the lab, up to a maximum mass of 200g.

8. Determine the spring constant by measuring the frequency of vertical oscillations and comparing to $T = 2\pi\sqrt{m/k}$. Use a 100 g mass for this part. In your report you should explain which result you believe gives you the more accurate value for k and why. This value should be used in subsequent calculations.

9. Replace the 100 g mass with a 50 g mass plus the taped group of washers labeled "collar". If the collar should happen to be missing, you can still obtain fairly good results using just the 50 g mass. Pull the mass down a few centimeters and release. You should observe the oscillations varying between a vertical mass and spring oscillation and a pendulum-like oscillation. Make sure to describe your observations in your Results section. After you stop the oscillation, measure the length of this oscillator as a pendulum (i.e., the distance from the pivot point to the center of mass of the 50+ g mass) and the mass of the collar. Calculate the frequency you would expect this system to oscillate at as a pendulum and compare it to the frequency you would calculate for it oscillating as a mass and spring system. If possible, measure this second frequency directly. [Note: the pendulum oscillation does not last long enough to time it directly.] You should find that the mass/spring frequency and the pendulum frequency differ by a factor of two for *this* hanging mass.

10. Use different values of masses while repeating the above. That is, for a 100 g hanging mass (no collar), determine the pendulum frequency and the mass/spring frequency and their ratio. Observe the oscillations, looking in particular for transition between spring-like and pendulum-like oscillations. Repeat using a 200 g hanging mass. What conclusion can you draw from these observations? Note that this is one piece of a much larger experiment in which you could show that you get these transitions between oscillation modes whenever the frequencies are related by an integral multiple.

11. When you have completed your measurements, please make sure the horizontal support bar is not positioned where someone is likely to run into it. Slide it up well above head level, if it is not already up that high.

What is due?

Cover page (with Abstract), Introduction, and Results section. As always, make sure to show uncertainties throughout all of your calculations. In the Introduction, you should not repeat word for word what I have written as an introduction to this portion of the lab manual. However, you do need to make sure that each equation you will use to analyze your data appears in your Introduction (not all of them appear in what I have written), and that the physics behind each equation is clear. Keep in mind that you are writing this report for someone with your background in physics but who has not taken this lab course. Use that as a guide in determining how much detail to provide and how many terms to define. Also remember to write the report as if your reader has not read the lab manual. You should make your lab report self-contained, without references to the lab manual.